

Advanced topology on the multiscale sequence spaces S^ν

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Abstract

We pursue the study of the multiscale spaces S^ν introduced by Jaffard in the context of multifractal analysis. We give the necessary and sufficient condition for S^ν to be locally p -convex, and exhibit a sequence of p -norms that defines its natural topology. The strong topological dual of S^ν is identified to another sequence space depending on ν , endowed with an inductive limit topology. As a particular case, we describe the dual of a countable intersection of Besov spaces.

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1 Introduction

The advent of multiscale analysis, for instance wavelet techniques, has shown the need for a new, hierarchical way of organizing information. Instead of a sequential data set $(u_n)_{n \in \mathbb{N}}$, one has often to work with a tree-indexed data set $(x_t)_{t \in T}$, where $T := \bigcup_{j \in \mathbb{N}_0} (\mathbb{Z}/\delta\mathbb{Z})^j$, $\delta \geq 2$ (without loss of generality, in all this paper we assume that $\delta = 2$).

In the long tradition of studying sequence spaces [6,11], no special interest was given to a hierarchical structure of the index set. But practical applications, such as multifractal analysis, rely heavily on this structure: coefficients at different scales j do not have the same importance, but within a same scale, they are often interchangeable. Sequence spaces emphasizing this specific feature have been introduced: for instance the classical Besov spaces, translated via the wavelet transform into Besov sequence spaces [14], and the intersections of Besov spaces [8]. However these spaces and their topologies (Besov-normed or projective limits thereof) provide only an indirect control, via weighted l^p sums, on the asymptotic repartition of the sizes of the coefficients. Direct control, such as asking that the number of coefficients having a certain size be bounded above, was the motivation which resulted in the more general class of topological vector spaces called S^ν .

Notations. In this paper we consider topological vector spaces (tvs) on the field \mathbb{C} of complex numbers. The set of strictly positive natural numbers is \mathbb{N} , and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. We write $a \wedge b$ and $a \vee b$ respectively for the minimum and the maximum of a and b . The integer parts are $\lfloor x \rfloor := \max \{z \in \mathbb{Z} : z \leq x\}$ and $\lceil x \rceil := \min \{z \in \mathbb{Z} : z \geq x\}$. The tree T is canonically identified to the set

of indices $\Lambda := \bigcup_{j \in \mathbb{N}_0} \{j\} \times \{0, \dots, 2^j - 1\}$; finally we define $\Omega := \mathbb{C}^\Lambda$ furnished with the pointwise convergence topology.

1.1 S^ν spaces

Inspired by the wavelet analysis of multifractal functions, Jaffard [9] introduced spaces S^ν of functions defined by conditions on their wavelet coefficients. It was shown that these spaces are *robust*, that is, they do not depend on the choice of the wavelet basis (provided the mother wavelet is sufficiently regular, localized, and has enough zero moments). So we can view them, and study their topology, as sequence spaces of wavelet coefficients; the set of indices Λ corresponds to taking the dyadic wavelet coefficients of a 1-periodic function.

In this context, ν is a non-decreasing right-continuous function of a real variable α , with values in $\{-\infty\} \cup [0, 1]$, that is not identically equal to $-\infty$ (an *admissible profile*). We define

$$\alpha_{\min} := \inf \{\alpha : \nu(\alpha) \geq 0\} \tag{1}$$

and

$$\alpha_{\max} := \inf \{\alpha : \nu(\alpha) = 1\} \tag{2}$$

with the habitual convention that $\inf \emptyset = +\infty$. It will be convenient to re-

member that this definition implies

$$\nu(\alpha) = -\infty \quad \text{if } \alpha < \alpha_{\min}$$

$$\nu(\alpha) \in [0, 1) \quad \text{if } \alpha_{\min} \leq \alpha < \alpha_{\max}$$

$$\nu(\alpha) = 1 \quad \text{if } \alpha \geq \alpha_{\max} \text{ (possibly this never happens).}$$

Definition 1 *The asymptotic profile of a sequence $x \in \Omega$ is*

$$\nu_x(\alpha) := \lim_{\varepsilon \rightarrow 0^+} \limsup_{j \rightarrow \infty} \frac{\log \left(\# \{k : |x_{j,k}| \geq 2^{-(\alpha+\varepsilon)j}\} \right)}{\log(2^j)} \quad (3)$$

It is easily seen that ν_x is always admissible.

Definition 2 *Given an admissible profile ν , we consider the vector space*

$$S^\nu := \left\{ x \in \Omega : \nu_x(\alpha) \leq \nu(\alpha) \quad \forall \alpha \in \mathbb{R} \right\}. \quad (4)$$

Heuristically, a sequence x belongs to S^ν if at each scale j , the number of k such that $|x_{j,k}| \geq 2^{-\alpha j}$ is of order $\leq 2^{\nu(\alpha)j}$.

Finally let us define the *concave conjugate* of ν is for $p > 0$

$$\eta(p) := \inf_{\alpha \geq \alpha_{\min}} (\alpha p - \nu(\alpha) + 1). \quad (5)$$

This function appears in the context of multifractal analysis in the so-called thermodynamic formalism [1,8].

1.2 Basic topology

Here we summarize the first properties of S^ν established in [3]: There exists a unique metrizable topology τ that is stronger than the pointwise convergence

and that makes S^ν a complete tvs. This topology is separable. If we define, for $\alpha \in \mathbb{R}$ and $\beta \in \{-\infty\} \cup [0, +\infty)$, the distance

$$d_{\alpha,\beta}(x, y) := d_{\alpha,\beta}(x - y) := \inf \left\{ C \geq 0 : \forall j, \# \left\{ |x_{j,k} - y_{j,k}| \geq C 2^{-\alpha j} \right\} \leq C 2^{\beta j} \right\} \quad (6)$$

(agreeing that $2^{j\beta} := 0$ when $\beta = -\infty$) and the ancillary metric space

$$E(\alpha, \beta) := \{x \in \Omega : d_{\alpha,\beta}(x) < \infty\} \quad (7)$$

then for any sequence α_n dense in \mathbb{R} and any sequence $\varepsilon_m \searrow 0$ we have

$$S^\nu = \bigcap_{n,m} E(\alpha_n, \nu(\alpha_n) + \varepsilon_m) \quad (8)$$

and τ coincides with the projective limit topology (the coarsest topology which makes each inclusion $S^\nu \subset E(\alpha_n, \nu(\alpha_n) + \varepsilon_m)$ continuous). Remark that $E(\alpha, \beta)$ is never a tvs when $0 \leq \beta < 1$, because the scalar multiplication is not continuous in $E(\alpha, \beta)$; however it is continuous in S^ν .

When $\beta = -\infty$, the space $E(\alpha, \beta)$ consists in the set of sequences x such that there exists C , for all $(j, k) \in \Lambda$, $|x_{j,k}| < C 2^{-\alpha j}$. This space corresponds to the space of wavelet coefficients of functions in the Hölder-Zygmund class C^α , as it was shown by Meyer [13].

Let us also recall the connexion with Besov spaces (Definition 5 in § 2.3 below).

If η is the concave conjugate of ν , then

$$S^\nu \subset \bigcap_{\varepsilon > 0} \bigcap_{p > 0} b_{p,\infty}^{\eta(p)/p-\varepsilon} \quad (9)$$

with equality if and only if ν is concave.

1.3 Prevalent properties

Some measure-related properties of S^ν : Given $(\psi_{j,k})$ an L^∞ -normalized orthogonal wavelet basis of $L^2(\mathbb{R}/\mathbb{Z})$ and assuming that $\alpha_{\min} > 0$, let $f_x := \sum_{j,k} x_{j,k} \psi_{j,k}$ and let $d_{f_x} : \alpha \mapsto \dim_H(\{t : h_{f_x}(t) = \alpha\})$ be its Hausdorff spectrum of Hölder singularities. Then for x in a prevalent [7] subset of S^ν , for all $\alpha \in \mathbb{R}$, we have $\nu_x(\alpha) = \nu(\alpha)$ and $d_{f_x}(\alpha) = \alpha \sup_{\alpha' \in (0, \alpha]} \frac{\nu(\alpha')}{\alpha'}$ if $\alpha \leq h_{\max}$, $d_{f_x}(\alpha) = -\infty$ else (here $h_{\max} := \inf_{\alpha > \alpha_{\min}} \frac{\alpha}{\nu(\alpha)}$). We refer to [2] for the details.

1.4 Outline of the results

Our goal in this paper is to establish advanced topological properties such as local convexity and duality for the spaces S^ν . We shall see that these properties, unlike those we recalled in § 1.2, depend in a subtle way on the profile ν .

In § 2 we give some definitions and preliminary results. In § 3 we study local convexity. Having shown that S^ν is never p -normable (§ 3.1), we give the necessary and sufficient condition for local p -convexity in § 3.2. A set of p -norms inducing the topology is presented in § 3.3. The last section (§ 4) of this article addresses the identification of the topological dual of S^ν . This dual $(S^\nu)'$ turns out (§ 4.2) to be a union of sequence spaces just smaller than another space $S^{\nu'}$, where ν' can be derived from ν in a way shown in § 4.1. In § 4.3 we prove that the strong topology on $(S^\nu)'$ is the same as the inductive limit topology on this union and deduce the condition for reflexivity. The particular case where S^ν is an intersection of Besov spaces is detailed in § 4.4.

2 Preliminaries

2.1 Right-inf derivative

We shall see that the local convexity of S^ν depends on ν , and more precisely on its right-inf derivative defined, whenever $\nu(\alpha) > -\infty$, as

$$\underline{\partial}^+ \nu(\alpha) := \liminf_{h \rightarrow 0^+} \frac{\nu(\alpha + h) - \nu(\alpha)}{h} \quad (10)$$

for which holds an equivalent of the mean value inequality.

Lemma 2.1 *Let ν be an admissible profile and $p > 0$. The following assertions are equivalent.*

- (1) *For all $\alpha_{\min} \leq \alpha < \alpha_{\max}$, $\underline{\partial}^+ \nu(\alpha) \geq p$;*
- (2) *For all $\alpha_{\min} \leq \alpha' \leq \alpha < \alpha_{\max}$, $\nu(\alpha) - \nu(\alpha') \geq p(\alpha - \alpha')$.*

Proof. The $2 \Rightarrow 1$ part is obvious. Conversely, let $f_n(x) := n(\nu(x + \frac{1}{n}) - \nu(x))$.

Note that $\liminf_n f_n(x) \geq \underline{\partial}^+ \nu(x) \geq p$ when $\alpha' \leq x \leq \alpha$. So

$$p(\alpha - \alpha') \leq \int_{\alpha'}^{\alpha} \liminf_n f_n(x) dx$$

By Fatou's lemma,

$$\begin{aligned} &\leq \liminf_n \int_{\alpha'}^{\alpha} f_n(x) dx \\ &\leq \liminf_n n \left(\int_{\alpha}^{\alpha + \frac{1}{n}} \nu(x) dx - \int_{\alpha'}^{\alpha' + \frac{1}{n}} \nu(x) dx \right) \\ &\leq \nu(\alpha) - \nu(\alpha') \end{aligned}$$

because ν is right-continuous. □

2.2 p -norm and local p -convexity

Let us recall a couple of definitions from Jarchow [10], which generalize the notions of norm and local convexity (these corresponding to the case $p = 1$).

Definition 3 *Let X be a vector space and $0 < p \leq 1$. A map $q : X \rightarrow [0, +\infty)$ is a p -seminorm if $q(\lambda x) = |\lambda|q(x)$ for all $\lambda \in \mathbb{C}$, $x \in X$ and if $q(x + y)^p \leq q(x)^p + q(y)^p$ for all $x, y \in X$.*

If in addition $q(x) = 0$ only if $x = 0$ then q is called a p -norm.

Definition 4 *Let $0 < p \leq 1$. A subset K of a vector space X is p -convex if for all $x_1, \dots, x_N \in K$ and $\theta_1, \dots, \theta_N$ such that $\sum_n \theta_n^p = 1$, the p -convex combination $\sum_n \theta_n x_n$ belongs to K .*

We say that K is absolutely p -convex if in addition it is circled (or balanced), that is, $\lambda x \in K$ whenever $x \in K$ and $|\lambda| \leq 1$.

The tvs X is locally p -convex if it has a basis of absolutely p -convex neighbourhoods of 0.

Clearly, a p -normed space is locally p -convex. For instance the sequence space l^p , $p > 0$, is $1 \wedge p$ -normed thus $1 \wedge p$ -locally convex.

When $X = \bigcap_n X_n$ endowed with the projective limit topology, X is locally p convex if and only if for each n , for each 0-neighbourhood V in X_n , there exist n_1, \dots, n_L and 0-neighbourhoods U_1, \dots, U_L in X_{n_1}, \dots, X_{n_L} such that any p -convex combination of elements of $U := \bigcap_l U_l$ stays in V .

2.3 Besov spaces

To end this section, and to prepare the results of the next one, we elucidate the link between (sequence) Besov spaces and the ancillary spaces $E(\alpha, \beta)$.

Definition 5 For $\alpha \in \mathbb{R}$, $0 < p < \infty$, the $b_{p,\infty}^\alpha$ Besov $1 \wedge p$ -norm of a sequence x is given by

$$\|x\|_{b_{p,\infty}^\alpha} := \sup_{j \in \mathbb{N}_0} 2^{(\alpha - \frac{1}{p})j} \left(\sum_{k=0}^{2^j-1} |x_{j,k}|^p \right)^{\frac{1}{p}} \quad (11)$$

and if $p = \infty$,

$$\|x\|_{b_{\infty,\infty}^\alpha} := \sup_{j \in \mathbb{N}_0} \sup_{0 \leq k < 2^j} 2^{\alpha j} |x_{j,k}|. \quad (12)$$

It is easy to check that, being modelled on l^p , the space $b_{p,\infty}^\alpha$ is $1 \wedge p$ -normed, thus $1 \wedge p$ -convex (and not better). Note that $b_{\infty,\infty}^\alpha = E(\alpha, -\infty)$.

Lemma 2.2 Let $0 < p < \infty$ and $s \in \mathbb{R}$. If $\beta \geq \alpha p + 1 - s$, then for all $x \in b_{p,\infty}^{s/p}$,

$$d_{\alpha,\beta}(x) \leq \|x\|_{b_{p,\infty}^{s/p}}^{\frac{p}{p+1}}. \quad (13)$$

Proof. Let $C := \|x\|_{b_{p,\infty}^{s/p}}$. If there exists a j such that

$$\#\{k : |x_{j,k}| \geq C^{\frac{p}{p+1}} 2^{-\alpha j}\} > C^{\frac{p}{p+1}} 2^{\beta j}$$

then $C \geq 2^{\frac{s-1}{p}j} (\sum_k |x_{j,k}|^p)^{\frac{1}{p}} > C 2^{\frac{\beta+s-1-\alpha p}{p}j} \geq C$, a flagrant contradiction. So it must be that for all j , $\#\{k : |x_{j,k}| \geq C^{\frac{p}{p+1}} 2^{-\alpha j}\} \leq C^{\frac{p}{p+1}} 2^{\beta j}$. This shows that $d_{\alpha,\beta}(x) \leq C^{\frac{p}{p+1}}$. \square

It follows that $b_{p,\infty}^{s/p} \subset E(\alpha, \beta)$ continuously, but the converse inclusion is never true.

Another Besov embedding will be useful.

Lemma 2.3 *If $0 < p \leq p'$ and $\alpha \in \mathbb{R}$, then for all $x \in b_{p',\infty}^\alpha$,*

$$\|x\|_{b_{p,\infty}^\alpha} \leq \|x\|_{b_{p',\infty}^\alpha}. \quad (14)$$

Proof. By Hölder's inequality, if $p \leq p'$,

$$\|x\|_{b_{p,\infty}^\alpha} = 2^{(\alpha - \frac{1}{p})j} \left(\sum_{k=0}^{2^j-1} |x_{j,k}|^p \right)^{\frac{1}{p}} \leq 2^{(\alpha - \frac{1}{p})j} 2^{(\frac{1}{p} - \frac{1}{p'})j} \left(\sum_{k=0}^{2^j-1} |x_{j,k}|^{p'} \right)^{\frac{1}{p'}} = \|x\|_{b_{p',\infty}^\alpha}.$$

□

Remark. This embedding uses specifically the fact that $0 \leq k2^{-j} < 1$ (Besov space on a compact domain). It can be compared to Lemma 8.2 of [3], which is valid on any domain (e.g. $k \in \mathbb{Z}$): When $p' \leq p$ and $\alpha - \frac{1}{p} \leq \alpha' - \frac{1}{p'}$ we also have $\|x\|_{b_{p,\infty}^\alpha} \leq \|x\|_{b_{p',\infty}^{\alpha'}}$.

3 Local geometry of S^ν

Here we apply the definitions of § 2.2 to S^ν spaces. We shall see that S^ν is never p -normable, but that it is locally p -convex for p depending on ν .

3.1 Non normability

Proposition 3.1 *The tvs S^ν is not p -normable for any $p > 0$.*

Proof. Suppose that q is a p -norm defining the topology of S^ν . Then there are $\alpha_l \in \mathbb{R}$, $\varepsilon_l > 0$ ($l = 1, \dots, L$) and $\delta_0 > 0$ such that

$$\bigcap_{l=1}^L U_l \subset B$$

where $B := \{x \in S^\nu : q(x) \leq 1\}$ and $U_l := \{x \in S^\nu : d_{\alpha_l, \nu(\alpha_l) + \varepsilon_l}(x) \leq \delta_0\}$. We assume $\alpha_1 < \dots < \alpha_L$ and $\delta_0 < 1$.

First case: There is l such that $\alpha_l < \alpha_{\min}$. Let n be the largest integer satisfying this. For all $l \leq n$, $\nu(\alpha_l) = -\infty$ and thus $d_{\alpha_l, \nu(\alpha_l) + \varepsilon_l}(x) = \sup_{(j,k) \in \Lambda} 2^{\alpha_l j} |x_{j,k}|$. For $m \in \mathbb{N}_0$ we define the sequence $x^m \in S^\nu$ by setting at scale m exactly one coefficient equal to $2^{-\alpha_n m} \delta_0$. We claim that $x^m \in B$ for sufficiently large m . Indeed, if $l \leq n$ then $d_{\alpha_l, \nu(\alpha_l) + \varepsilon_l}(x^m) \leq \delta_0$ for all m ; if $l > n$ we have $\nu(\alpha_l) + \varepsilon_l > 0$ hence $d_{\alpha_l, \nu(\alpha_l) + \varepsilon_l}(x^m) \leq \delta_0$ as soon as $m \geq -\frac{\log_2(\delta_0)}{\nu(\alpha_l) + \varepsilon_l}$. So $x^m \in \bigcap_{l=1}^L U_l \subset B$.

Let us now consider $\alpha_n < \alpha' < \alpha_{\min}$. If B' denotes the unit ball in $E(\alpha', -\infty)$, our hypothesis that the topology of S^ν stems from the p -norm q implies that there exists $\lambda > 0$ such that $\lambda B \subset B'$. This would imply that $\lambda x^m \in B'$ for all m , a contradiction because $d_{\alpha', -\infty}(x^m) = 2^{(\alpha' - \alpha_n)m} \rightarrow \infty$.

Second case: $\alpha_l \geq \alpha_{\min}$ for all l . We chose $\alpha'' < \alpha' < \alpha_{\min}$ and define the sequence $x^m \in S^\nu$ by setting at scale m exactly one coefficient equal to $2^{-\alpha'' m} \delta_0$; the rest of the proof is identical to the sub-case $l > n$ above. \square

3.2 Local convexity

The convexity index will be

$$p_0 := 1 \wedge \inf_{0 \leq \nu(\alpha) < 1} \underline{\partial}^+ \nu(\alpha) \quad (15)$$

(we recall that $0 \leq \nu(\alpha) < 1$ is equivalent to $\alpha_{\min} \leq \alpha < \alpha_{\max}$).

How does this number come into play? When ν is concave, we know from (9) that S^ν is an intersection of (sequence) Besov spaces $b_{p,\infty}^{\eta(p)/p-\varepsilon}$ for $\varepsilon > 0$ and

$p > 0$, η being the convex conjugate (5) of ν . Suppose that $0 < p \leq p_0$, and observe that by Lemma 2.1, $\frac{\eta(p)}{p} = \frac{\eta(p_0)}{p_0} (= \alpha_{\max}) < \infty$. Then Lemma 2.3 leads to $b_{p,\infty}^{\eta(p)/p-\varepsilon} \supset b_{p_0,\infty}^{\eta(p_0)/p_0-\varepsilon}$. So in fact

$$S^\nu = \bigcap_{\varepsilon > 0} \bigcap_{p \geq p_0} b_{p,\infty}^{\eta(p)/p-\varepsilon}$$

an intersection of spaces at least p_0 -convex. This idea leads to the general case.

Theorem 1 *Let p_0 be defined by (15)*

- *If $p_0 > 0$, then S^ν is locally p_0 -convex.*
- *If $p_0 < 1$, then S^ν is not locally p -convex for any $p > p_0$.*

Proof. We first prove the second point. Suppose that $p_0 < p < 1$. Our purpose is to find a neighbourhood V in S^ν , for instance the unit ball in some $E(\alpha', \nu(\alpha') + \varepsilon)$, such that it cannot contain the p -convex hull of any 0-neighbourhood U .

By definition of p_0 , there exist $\varepsilon > 0$ and $\alpha_{\min} \leq \alpha < \alpha'$ such that $\nu(\alpha') + \varepsilon < 1$ and $\nu(\alpha') - \nu(\alpha) + \varepsilon < p(\alpha' - \alpha)$. For short we shall write

$$s := \alpha' - \alpha$$

and

$$t := \nu(\alpha') - \nu(\alpha) + \varepsilon.$$

Thanks to the right-continuity of ν , ε and s can be taken small enough so that

$$\frac{p}{p+1}(s+t) < 1 - \nu(\alpha). \quad (16)$$

Assume that U is a p -convex 0-neighbourhood in S^ν such that

$$U \subset V = \left\{ x \in S^\nu : d_{\alpha', \nu(\alpha') + \varepsilon}(x) < 1 \right\}$$

and let $z \in S^\nu$ be such that $z_{j,k} = 2^{-\alpha j}$ for $\lfloor 2^{\nu(\alpha)j} \rfloor$ values of k at each scale j . Since S^ν is a tvs, there is $\lambda > 0$ such that $\lambda z \in U$; moreover, the special structure of the topology of S^ν allows to say that this remains true if some coefficients have been set to 0 or moved within the scale.

Now, let $N \in \mathbb{N}$ be fixed. If j_0 is the smallest integer such that $2^{j_0} \geq N2^{\nu(\alpha)j_0}$, we construct x_1, \dots, x_N having, at each scale $j \geq j_0$ disjoint sets of cardinal $\lfloor 2^{\nu(\alpha)j} \rfloor$ of coefficients equal to $\lambda 2^{-\alpha j}$ (and the others are 0). These sequences all belong to U . We then form the p -convex combination

$$x := N^{-\frac{1}{p}} \sum_{n=1}^N x_n.$$

Note that, at each scale $j \geq j_0$, the sequence x has $N \lfloor 2^{\nu(\alpha)j} \rfloor = C'(N, j) 2^{\nu(\alpha')j}$ coefficients of size equal to $N^{-\frac{1}{p}} \lambda 2^{-\alpha j} = C(N, j) 2^{-\alpha' j}$, with $C(N, j) := \lambda N^{-\frac{1}{p}} 2^{sj}$ and $C'(N, j) := N 2^{-tj} \frac{\lfloor 2^{\nu(\alpha)j} \rfloor}{2^{\nu(\alpha)j}}$.

Let us focus on the scale $j := \left\lceil \frac{\frac{p+1}{p} \log_2(N) - \log_2(\lambda)}{s+t} \right\rceil$. Because of (16), we check that this $j \geq j_0$. Furthermore since $j \geq \frac{\frac{p+1}{p} \log_2(N) - \log_2(\lambda)}{s+t}$,

$$\begin{aligned} C(N, j) &\geq \lambda N^{-\frac{1}{p}} 2^{\frac{s}{s+t} \left(\frac{p+1}{p} \log_2(N) - \log_2(\lambda) \right)} \\ &\geq \lambda^{\frac{t}{s+t}} N^{\frac{ps-t}{p(s+t)}} \end{aligned}$$

and since $j < 1 + \frac{\frac{p+1}{p} \log_2(N) - \log_2(\lambda)}{s+t}$,

$$\begin{aligned} C'(N, j) &> \frac{N}{2} 2^{-t - \frac{t}{s+t} \left(\frac{p+1}{p} \log_2(N) - \log_2(\lambda) \right)} \\ &> 2^{-t-1} \lambda^{\frac{t}{s+t}} N^{\frac{ps-t}{p(s+t)}}. \end{aligned}$$

By the hypothesis on p , the exponent of N is strictly positive and this shows that $d_{\alpha', \nu(\alpha') + \varepsilon}(x)$ can be arbitrarily large with N . In particular, no matter how small the neighbourhood U is, there exists a p -convex combination of elements of U which does not belong to V , meaning the unit ball in $E(\alpha', \nu(\alpha') + \varepsilon)$.

So S^ν is not locally p -convex.

The proof of the first point boils down to this: Let $M > 0$, $\varepsilon > 0$ and α be fixed, we want to find a 0-neighbourhood U in S^ν such that any p_0 -convex combination of elements $x_1, \dots, x_N \in U$ will stay in $V := \{x \in S^\nu : d_{\alpha, \nu(\alpha) + \varepsilon}(x) \leq M\}$.

The two cases $\nu(\alpha) = -\infty$ and $\nu(\alpha) = 1$ are both trivial because $E(\alpha, -\infty) = b_{\infty, \infty}^\alpha$ and $E(\alpha, 1 + \varepsilon) = \Omega$ (the set of all sequences with the topology of pointwise convergence, see [4]) are locally convex.

From now on we assume that $0 \leq \nu(\alpha) < 1$. Let $L := \left\lceil (\alpha - \alpha_{\min}) \frac{2p_0}{\varepsilon} \right\rceil$, $\lambda := \frac{M}{L+2}$ and for $-1 \leq l \leq L$ let $\alpha_l := \alpha_{\min} + \frac{\varepsilon}{2p_0}l$ and $\nu_l := \nu(\alpha_l) + \frac{\varepsilon}{2}$. Note that $\alpha_L \geq \alpha$. We construct

$$U := \bigcap_{l=-1}^L U_l \quad (17)$$

where

$$U_l := \{x \in S^\nu : d_{\alpha_l, \nu_l}(x) < \lambda\}$$

(in particular, since $\nu_{-1} = -\infty$, $U_{-1} \subset \{x : \forall j, k, |x_{j,k}| < \lambda 2^{-\alpha_{-1}j}\}$).

For an arbitrary $N \in \mathbb{N}$, let $x_1, \dots, x_N \in U$ and $\theta_1, \dots, \theta_N$ be the coefficients of a p_0 -convex combination $x := \sum_{n=1}^N \theta_n x_n$. We split each x_n as $x_n = \sum_{l=0}^{L+1} x_n^l$, where for $0 \leq l \leq L$, x_n^l receives the coefficients $\lambda 2^{-\alpha_l j} < |x_{j,k}| \leq \lambda 2^{-\alpha_{l-1} j}$ (the others are set to 0) and x_n^{L+1} receives the coefficients $|x_{j,k}| \leq \lambda 2^{-\alpha_L j}$; since $x_n \in U_{-1}$ it has no coefficients $|x_{j,k}| > \lambda 2^{-\alpha_{-1} j}$. Once this is done, we do the p_0 -convex combinations

$$x^l := \sum_{n=1}^N \theta_n x_n^l.$$

Remark that $x_n \in U$ implies that each $x_n^l \in U$ and a fortiori $x_n^l \in U_l$. Let us contemplate two cases.

When $0 \leq l \leq L$: Since x_n^l is in U_l and has only coefficients $|x_{j,k}| > \lambda 2^{-\alpha_l j}$,

the cardinal of the set of non-zero coefficients at scale j in x_n^l is smaller than $\lambda 2^{\nu_l j}$, and these coefficients are all bounded from above by $\lambda 2^{-\alpha_{l-1} j}$. It follows that if $s - 1 + \nu_l - \alpha_{l-1} p \leq 0$, then $\|x_n^l\|_{b_{p,\infty}^{s/p}} \leq \lambda^{\frac{p+1}{p}}$. Actually we take $p = p_0$ and $s = s_l := 1 + \alpha_{l-1} p_0 - \nu_l$. Since $b_{p_0,\infty}^{s_l/p_0}$ is p_0 -convex, $\|x^l\|_{b_{p_0,\infty}^{s_l/p_0}} \leq \lambda^{\frac{p_0+1}{p_0}}$ as well.

Thanks to Lemma 2.1 and the definition of p_0 ,

$$\begin{aligned} \alpha p_0 + 1 - s_l - \nu(\alpha) - \varepsilon &= p_0(\alpha - \alpha_{l-1}) + \nu_l - \nu(\alpha) - \varepsilon \\ &= p_0(\alpha - \alpha_l + \frac{\varepsilon}{2p_0}) + \nu(\alpha_l) + \frac{\varepsilon}{2} - \nu(\alpha) - \varepsilon \\ &= p_0(\alpha - \alpha_l) + \nu(\alpha_l) - \nu(\alpha) \\ &\leq 0. \end{aligned}$$

Thus Lemma 2.2 is applicable with $\beta = \nu(\alpha) + \varepsilon$, and we get that

$$d_{\alpha, \nu(\alpha) + \varepsilon}(x^l) \leq \|x^l\|_{b_{p_0,\infty}^{\frac{p_0}{p_0+1}}} \leq \lambda.$$

When $l = L + 1$, simply notice that what remains in each x_n^{L+1} are coefficients $|x_{j,k}| \leq \lambda 2^{-\alpha_j}$. The p_0 -convex combination x^{L+1} will have *a fortiori* only coefficients $|x_{j,k}| \leq \lambda 2^{-\alpha_j}$ and this shows that $d_{\alpha, \nu(\alpha) + \varepsilon}(x) \leq \lambda$ (indeed any $C > \lambda$ satisfies the condition in (6), so their infimum is $\leq \lambda$).

Finally,

$$d_{\alpha, \nu(\alpha) + \varepsilon}(x) \leq \sum_{l=0}^{L+1} d_{\alpha, \nu(\alpha) + \varepsilon}(x^l) \leq (L+2)\lambda = M.$$

We have proved that S^ν is locally p_0 -convex.

□

Corollary 3.2 *The space S^ν is a Fréchet space if and only if $\underline{\partial}^+ \nu(\alpha) \geq 1$ for all $\alpha \in [\alpha_{\min}, \alpha_{\max})$.*

Proof. We already know that S^ν is a metrizable and complete tvs [3]. We apply the previous theorem with $p_0 = 1$. \square

3.3 Another description of the topology

The topology of a Fréchet space can always be defined by a sequence of seminorms. When the space is only p_0 -convex, the seminorms are naturally to be replaced by p_0 -seminorms. The interesting feature in the case of S^ν is that they can be made explicit as p_0 -norms interpolating two Besov spaces.

Theorem 2 *If $p_0 > 0$, the topology of S^ν is induced by the family of norms $\|x\|_{b_{\infty,\infty}^{\alpha_{\min}-\varepsilon}}$ together with the p_0 -norms*

$$\|x\|_{\alpha,\varepsilon} := \inf \left\{ \|x'\|_{b_{p_0,\infty}^s} + \|x''\|_{b_{\infty,\infty}^\alpha} : x' + x'' = x \right\} \quad (18)$$

where $\alpha \in [\alpha_{\min}, \alpha_{\max})$, $\varepsilon > 0$ and $s := \alpha + \frac{1-\nu(\alpha)}{p_0} - \varepsilon$.

This family of p_0 -norms can be made countable by taking sequences (α_n) dense in $[\alpha_{\min}, \alpha_{\max})$ and $(\varepsilon_n) \rightarrow 0^+$.

To illustrate this theorem, notice (Figure 1) that the hypograph of ν is the intersection of the sets (parameters domains) of $(\tilde{\alpha}, \beta)$ such that a sequence having $2^{\beta j}$ coefficients $= 2^{-\tilde{\alpha} j}$ belongs to $b_{p_0,\infty}^s + b_{\infty,\infty}^\alpha$.

Proof. Let $\alpha \in [\alpha_{\min}, \alpha_{\max})$ and $\varepsilon > 0$. As in the proof of the first point of Theorem 1, we define L , α_l and ν_l for $-1 \leq l \leq L-1$ and $\alpha_L := \alpha$, $\nu_L := \nu(\alpha_L) + \frac{\varepsilon}{2}$. Let $0 < \delta < 1$ be fixed and let $\lambda := \frac{\delta}{L+2}$. The neighbourhoods U_l and, for $x \in \bigcap_{l=-1}^L U_l$, the splitting $x = \sum_{l=0}^{L+1} x^l$ are unchanged.

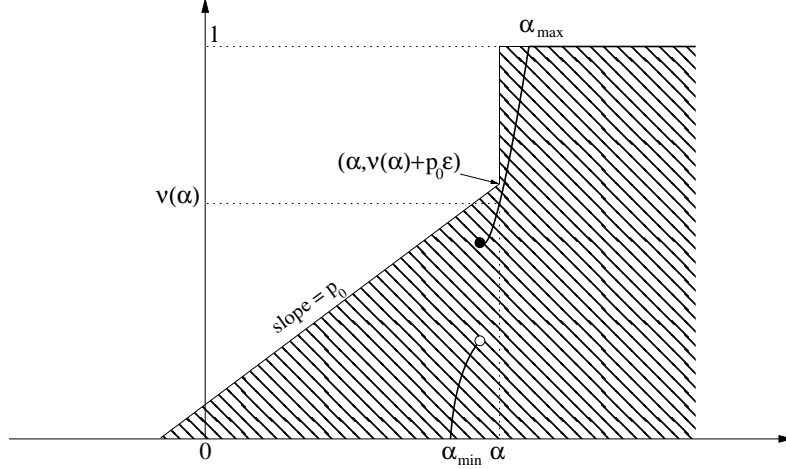


Fig. 1. $\nu(\alpha)$ and the parameters domain of $b_{p_0,\infty}^s + b_{\infty,\infty}^\alpha$

It is clear that, since $\alpha_L \geq \alpha$, $x'' := x^{L+1}$ belongs to $b_{\infty,\infty}^\alpha$ and moreover $\|x''\|_{b_{\infty,\infty}^\alpha} \leq \lambda$. On the other hand, with $s := \alpha + \frac{1-\nu(\alpha)}{p_0} - \varepsilon$, for $0 \leq l \leq L$, each x^l belongs to $b_{p_0,\infty}^s$ and $\|x^l\|_{b_{p_0,\infty}^s} \leq \lambda^{\frac{p_0+1}{p_0}} \leq \lambda$. So finally $x' := \sum_{l=0}^L x^l$ satisfies $\|x'\|_{b_{p_0,\infty}^s} \leq (L+1)\lambda$. We have proved that by our choice of λ ,

$$\bigcap_{l=-1}^L U_l \subset \{x \in S^\nu : \|x\|_{\alpha,\varepsilon} \leq \delta\}$$

in other words, that $x \mapsto \|x\|_{\alpha,\varepsilon}$ is a continuous p_0 -norm on S^ν .

Now, let us show that given $\alpha \in [\alpha_{\min}, \alpha_{\max})$, $\varepsilon > 0$ and $0 < \delta < 1$, we have

$$\left\{ x \in S^\nu : \|x\|_{\alpha,\varepsilon} < \left(\frac{\delta}{2}\right)^{\frac{p_0+1}{p_0}} \right\} \subset \left\{ x \in S^\nu : d_{\alpha,\nu(\alpha)+\varepsilon}(x) \leq \delta \right\}.$$

If $\|x\|_{\alpha,\varepsilon} < \left(\frac{\delta}{2}\right)^{\frac{p_0+1}{p_0}}$, then there exist $x', x'' \in S^\nu$ such that $x = x' + x''$, $\|x'\|_{b_{p_0,\infty}^s} < \left(\frac{\delta}{2}\right)^{\frac{p_0+1}{p_0}}$ and $\|x''\|_{b_{\infty,\infty}^\alpha} < \left(\frac{\delta}{2}\right)^{\frac{p_0+1}{p_0}} \leq \frac{\delta}{2}$.

Using the inequality between distances $d_{\alpha,\beta}$ and Besov p -norms (Lemma 2.2)

and the fact that $\sup_{j,k} 2^{\alpha j} |x''_{j,k}| \leq \frac{\delta}{2}$ implies $d_{\alpha,\nu(\alpha)+\varepsilon}(x'') \leq \frac{\delta}{2}$, we get

$$d_{\alpha,\nu(\alpha)+\varepsilon}(x) \leq d_{\alpha,\nu(\alpha)+\varepsilon}(x') + d_{\alpha,\nu(\alpha)+\varepsilon}(x'') \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

and we are done. Thanks to (8) we see that if $\alpha \in (\alpha_n)$ describes a dense set in $[\alpha_{\min}, \alpha_{\max})$ and if $\varepsilon \in (\varepsilon_m)$ has limit 0, together with the $b_{\infty, \infty}^{\alpha_{\min} - \varepsilon}$ norms, these p_0 -norms completely define the topology of S^ν . \square

4 Duality

We employ the usual scalar product on the sequence space Ω

$$\langle x, y \rangle := \sum_{(j,k) \in \Lambda} x_{j,k} \overline{y_{j,k}} \quad (19)$$

to identify the topological dual $(S^\nu)'$ of S^ν to a sequence space that will be revealed by Theorem 3. This identification is made as follows: for all $(j, k) \in \Lambda$, let $e^{j,k}$ be the sequence whose only non-zero component is $e_{j,k}^{j,k} = 1$. Given $u \in (S^\nu)'$, let us define

$$y := \sum_{j,k \in \Lambda} \overline{u(e^{j,k})} e^{j,k}.$$

This sequence y indeed satisfies $u(x) = \langle x, y \rangle$ because for all $x \in S^\nu$, the sum $\sum_{j,k \in \Lambda} x_{j,k} e^{j,k}$ converges to x in S^ν (see [3]).

Remark. If we keep in mind that $(x_{j,k})$ represents the wavelet coefficients of a function, the above scalar product corresponds to the L^2 scalar product if and only if the said wavelets form an orthonormal basis of $L^2([0, 1])$. In [2,4,5], we use L^∞ -normalized orthogonal wavelets instead (which is more convenient when one uses wavelets to study the pointwise regularity of functions); in that setting, the L^2 product of functions corresponds to the coefficients product

$$\langle\langle x, y \rangle\rangle := \sum_{(j,k) \in \Lambda} 2^{-j} x_{j,k} \overline{y_{j,k}}. \quad (20)$$

The results in this section translate easily in terms of the duality $\langle\langle \cdot, \cdot \rangle\rangle$ by shifting the symmetry axis in Figure 2 by $1/2$ to the left.

4.1 Dual profile

Let us fix a few notations. We shall write

$$\llbracket \beta \rrbracket := \begin{cases} -\infty & \text{if } \beta < 0 \\ \beta & \text{if } 0 \leq \beta \leq 1 \\ 1 & \text{if } \beta \geq 1. \end{cases} \quad (21)$$

Definition 6 *The dual profile of ν is the function*

$$\nu' : \alpha' \mapsto \llbracket \alpha' + \inf \{ \alpha : \nu(\alpha) - \alpha > \alpha' \} \rrbracket. \quad (22)$$

As in (1) and (2) we define the corresponding α'_{\min} and α'_{\max} .

It is easily seen that $\alpha'_{\min} = -\alpha_{\min}$ and that $1 - \alpha_{\max} \leq \alpha'_{\max} \leq 1 - \alpha_{\min}$. Graphically, except for the discontinuities and the part where the value 1 is attained, the graph of ν' is obtained by horizontal symmetry, with respect to the axis $\beta = 2\alpha$, of the graph of ν (Figure 2). Discontinuities in ν correspond to zones with slope 1 in ν' ; zones with slope ≤ 1 in ν correspond to right-continuous discontinuities in ν' (see Proposition 4.1 below).

In the next proposition we state the less obvious properties that will be useful to us.

Proposition 4.1 *The dual profile ν' of ν verifies*

- (i) ν' is right-continuous;

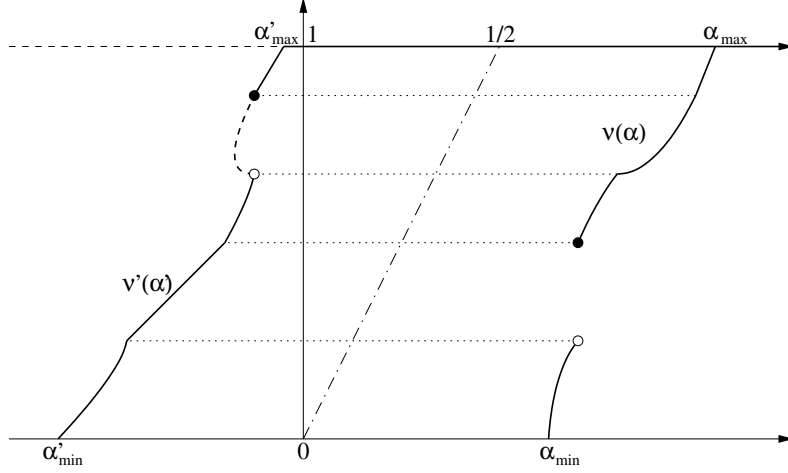


Fig. 2. The symmetry between ν and ν'

- (ii) for all α, α' , $\alpha + \alpha' \geq \nu(\alpha) \wedge \nu'(\alpha')$;
- (iii) if $\nu'(\alpha') = \alpha + \alpha'$ then $\nu'(\alpha') \leq \nu(\alpha)$;
- (iv) ν' is non-decreasing; furthermore for all $\alpha' \leq \alpha'_{\max}$ and $\varepsilon \geq 0$,

$$\nu'(\alpha' - \varepsilon) \leq \nu'(\alpha') - \varepsilon.$$

Proof. We can suppose $\alpha' \in [\alpha'_{\min}, \alpha'_{\max}]$, as the other cases are trivial.

(i): Let α' be fixed. For all $\varepsilon > 0$, there exists an α such that $\alpha' < \nu(\alpha) - \alpha$ and $\nu'(\alpha') + \varepsilon \geq \alpha + \alpha'$. Then with $\eta := \varepsilon \wedge (\nu(\alpha) - (\alpha + \alpha')) > 0$, $\zeta' < \alpha' + \eta$ implies that $\zeta' < \nu(\alpha) - \alpha$ as well, so $\nu'(\zeta') \leq \zeta' + \alpha < \alpha' + \alpha + \eta \leq \nu'(\alpha') + 2\varepsilon$. This proves right-continuity at α' .

(ii): Given α and α' , if $\nu(\alpha) > \alpha + \alpha'$ then $\alpha \geq \inf \{\tilde{\alpha} : \nu(\tilde{\alpha}) > \tilde{\alpha} + \alpha'\}$ hence $\alpha + \alpha' \geq \nu'(\alpha')$; otherwise $\alpha + \alpha' \geq \nu(\alpha)$.

(iii): If $\nu'(\alpha') = \alpha + \alpha'$ then $\alpha = \inf \{\tilde{\alpha} : \nu(\tilde{\alpha}) > \tilde{\alpha} + \alpha'\}$ thus by right-continuity $\nu(\alpha) \geq \alpha + \alpha' = \nu'(\alpha')$.

(iv) is trivial. □

4.2 Topological dual of S^ν

For $\varepsilon > 0$ we write $\nu'_\varepsilon(\alpha') := \nu'(\alpha' - \varepsilon)$.

Theorem 3 *The topological dual of S^ν is*

$$(S^\nu)' = \bigcup_{\varepsilon > 0} S^{\nu'_\varepsilon}. \quad (23)$$

Proof. Suppose first that $y \notin S^{\nu'_\varepsilon}$ for any $\varepsilon > 0$ and let us construct an $x \in S^\nu$ such that $\langle x, y \rangle = \infty$. The hypothesis on y implies that, for every $\varepsilon > 0$, there exist $\alpha' \in \mathbb{R}$, $\delta > 0$ such that $y \notin E(\alpha', \nu'_\varepsilon(\alpha') + \delta)$; in particular, y does not belong to any of the balls of this space. So, given any strictly positive sequence $\varepsilon_n \rightarrow 0$, we thus construct sequences of reals $(\alpha'_n)_{n \in \mathbb{N}}$ and integers $(j_n)_{n \in \mathbb{N}}$ (the latter we can make strictly increasing) such that for all $n \in \mathbb{N}$,

$$\#\{k : |y_{j_n, k}| \geq 2^{-\alpha'_n j_n}\} > 2^{\nu'(\alpha'_n - \varepsilon_n) j_n}. \quad (24)$$

Remark that it is very possible that for some n , $\nu'(\alpha'_n - \varepsilon_n) = -\infty$, which is equivalent to

$$\alpha'_n - \varepsilon_n < \alpha'_{\min} = -\alpha_{\min}; \quad (25)$$

we denote by I the set of such indices and by $J := \mathbb{N} \setminus I$ its complement. When $n \in J$, we remark that we have

$$\nu'(\alpha'_n - \varepsilon_n) = \alpha'_n - \varepsilon_n + \inf \{\alpha : \nu(\alpha) - \alpha > \alpha'_n - \varepsilon_n\} \geq \alpha'_n - \varepsilon_n + \alpha_{\min}. \quad (26)$$

To construct x we put

- for all $n \in J$, at $\left\lceil 2^{\nu'(\alpha'_n - \varepsilon_n) j_n} \right\rceil$ of the positions where $|y_{j_n, k}| \geq 2^{-\alpha'_n j_n}$,

$$x_{j_n, k} := 2^{-\alpha_n j_n} \frac{y_{j_n, k}}{|y_{j_n, k}|}$$

with $\alpha_n := \nu'(\alpha'_n - \varepsilon_n) - \alpha'_n$;

- for all $n \in I$, at exactly one position where $|y_{j_n,k}| \geq 2^{-\alpha'_n j_n}$ we put $x_{j_n,k}$ having the same expression as above, but with $\alpha_n := -\alpha'_n$;

and naturally all the other coefficients of x are set equal to 0.

The scalar product $\langle x, y \rangle$ is divergent because for all $n \in \mathbb{N} \setminus I = J$

$$\sum_{0 \leq k < 2^{j_n}} x_{j_n,k} \overline{y_{j_n,k}} \geq 2^{\nu'(\alpha'_n - \varepsilon_n)j_n} 2^{(\alpha'_n - \nu'(\alpha'_n - \varepsilon_n))j_n} 2^{-\alpha'_n j_n} = 1$$

whereas if $n \in I$,

$$\sum_{0 \leq k < 2^{j_n}} x_{j_n,k} \overline{y_{j_n,k}} = 2^{\alpha'_n j_n} 2^{-\alpha'_n j_n} = 1.$$

It remains to prove that x belongs to $S^\nu = \bigcap_{\alpha \in \mathbb{R}, \varepsilon > 0} E(\alpha, \nu(\alpha) + \varepsilon)$. For this we consider two cases.

Firstly, if $\alpha < \alpha_{\min}$: In this situation, we have to show that $\sup_{j,k} 2^{\alpha j} |x_{j,k}| < \infty$ or, equivalently, that

$$(i) \quad \sup_{n \in I} 2^{\alpha j_n} 2^{\alpha'_n j_n} < \infty \quad \text{and} \quad (ii) \quad \sup_{n \in J} 2^{\alpha j_n} 2^{(\alpha'_n - \nu'(\alpha'_n - \varepsilon_n))j_n} < \infty.$$

When $n \in I$, by (25) we have $\alpha'_n + \alpha < \varepsilon_n + \alpha - \alpha_{\min}$, whereas when $n \in J$, by (26) we get $\alpha + \alpha'_n - \nu'(\alpha'_n - \varepsilon_n) \leq \varepsilon_n + \alpha - \alpha_{\min}$. Since $\varepsilon_n \rightarrow 0$ this quantity becomes negative when n is large enough, so both (i) and (ii) hold.

Secondly, if $\alpha \geq \alpha_{\min}$: In that case, let $\beta := \nu(\alpha) + \varepsilon$. We have to show that there exists $C > 0$ such that for all n ,

$$\#\{k : |x_{j_n,k}| \geq C 2^{-\alpha j_n}\} \leq C 2^{\beta j_n}.$$

When $n \in I$ this is trivial, since $\beta > 0$ and there is only one non-zero coefficient in x at scale j_n . When $n \in J$, either $\alpha_n = \nu'(\alpha'_n - \varepsilon_n) - \alpha'_n > \alpha$, and the above

cardinal is zero, or $\nu'(\alpha'_n - \varepsilon_n) - \alpha'_n \leq \alpha$. In that last case, using the right-continuity of ν we get that $\nu(\alpha_n + \varepsilon_n) \leq \beta$ for n large enough; using (iii) of Proposition 4.1 we have $\nu'(\alpha'_n - \varepsilon_n) \leq \nu(\alpha_n + \varepsilon_n) \leq \beta$ and the conclusion follows. We have proved that $y \notin \bigcup_{\varepsilon>0} S^{\nu'_\varepsilon}$ cannot belong to the dual of S^ν .

Conversely, let $\varepsilon > 0$ and $y \in S^{\nu'_{2\varepsilon}}$. We construct $L := \left\lceil \frac{4}{\varepsilon} \right\rceil$ and for $-1 \leq l \leq L$, $\alpha_l := \alpha_{\min} + \frac{\varepsilon}{4}l$ and $\nu_l := \nu(\alpha_l) + \frac{\varepsilon}{4}$. Similarly, for $-1 \leq l' \leq L$, $\alpha'_{l'} := \alpha'_{\min} + 2\varepsilon + \frac{\varepsilon}{4}l'$ and $\mu_{l'} := \nu'(\alpha'_{l'} - 2\varepsilon) + \frac{\varepsilon}{4}$.

Let $U := \bigcap_{l=-1}^L U_l$, where U_l is the open unit ball in $E(\alpha_l, \nu_l)$ and fix an $A > \max_{-1 \leq l' \leq L} d_{\alpha'_{l'}, \mu_{l'}}(y)$. We split any $x \in U$ as $x = \sum_{l=0}^{L+1} x_l$, where for $0 \leq l \leq L$, x_l receives the coefficients $2^{-\alpha_l j} < |x_{j,k}| \leq 2^{-\alpha_{l-1} j}$ and x_{L+1} receives the coefficients $|x_{j,k}| \leq 2^{-\alpha_L j}$ (since $\nu(\alpha_{-1}) = -\infty$ and $x \in U_{-1}$, there is no coefficient $|x_{j,k}| > 2^{-\alpha_{-1} j}$).

We do the same to y , writing $y = \sum_{l'=0}^{L+1} y_{l'}$, where for $0 \leq l' \leq L$, $y_{l'}$ receives the coefficients $A2^{-\alpha'_{l'} j} < |y_{j,k}| \leq A2^{-\alpha'_{l'-1} j}$ and y_{L+1} receives the coefficients $|y_{j,k}| \leq A2^{-\alpha'_{L+1} j}$ (same remark as above about the coefficients $|y_{j,k}| > A2^{-\alpha'_{-1} j}$: there are none because $\mu_{-1} = \nu'(\alpha'_{\min} - \frac{\varepsilon}{4}) = -\infty$).

The proof now boils down to studying each term of

$$\langle x, y \rangle = \sum_{l=0}^{L+1} \sum_{l'=0}^{L+1} \langle x_l, y_{l'} \rangle. \quad (27)$$

Four cases can be distinguished.

Firstly, if $0 \leq l \leq L$, $0 \leq l' \leq L$ and $\nu(\alpha_l) \leq \nu'(\alpha'_{l'} - \varepsilon)$: Then by Proposition 4.1 (ii) applied to α_l and $\alpha'_{l'} - \varepsilon$, this means that $\alpha_l + \alpha'_{l'} \geq \nu(\alpha_l) + \varepsilon = \nu_l + \frac{3\varepsilon}{4}$. At scale j in x_l , there are less than $2^{\nu_l j}$ non-zero coefficients which are bounded

by $2^{-\alpha_{l-1}j}$, so

$$|\langle x_l, y_{l'} \rangle| \leq A \sum_{j \in \mathbb{N}_0} 2^{(\nu_l - \alpha_{l-1} - \alpha'_{l'-1})j} \leq A \sum_{j \in \mathbb{N}_0} 2^{-\frac{\varepsilon}{4}j}.$$

Secondly, if $0 \leq l \leq L$, $0 \leq l' \leq L$ and $\nu(\alpha_l) \geq \nu'(\alpha'_{l'} - \varepsilon)$: Then by Proposition 4.1 (ii) once again we get $\alpha_l + \alpha'_{l'} \geq \nu(\alpha_l) \wedge \nu'(\alpha'_{l'}) \geq \nu'(\alpha'_{l'} - \varepsilon) \geq \nu'(\alpha'_{l'} - 2\varepsilon) + \varepsilon = \mu_{l'} + \frac{3\varepsilon}{4}$. At scale j , in $y_{l'}$ there are less than $A2^{\mu_{l'}j}$ non-zero coefficients which are bounded by $A2^{-\alpha'_{l'-1}j}$, so

$$|\langle x_l, y_{l'} \rangle| \leq A^2 \sum_{j \in \mathbb{N}_0} 2^{(\mu_{l'} - \alpha_{l-1} - \alpha'_{l'-1})j} \leq A^2 \sum_{j \in \mathbb{N}_0} 2^{-\frac{\varepsilon}{4}j}.$$

Thirdly, if $l = L + 1$ and $0 \leq l' \leq L + 1$: Since $\alpha'_{l'-1} \geq -\alpha_{\min} + \frac{7\varepsilon}{4}$ and $\alpha_L \geq \alpha_{\min} + 1$, we get by a direct computation

$$|\langle x_{L+1}, y_{l'} \rangle| \leq A \sum_{j \in \mathbb{N}_0} 2^{(1 - \alpha_L - \alpha'_{l'-1})j} \leq A \sum_{j \in \mathbb{N}_0} 2^{-\frac{7\varepsilon}{4}j}.$$

Finally, if $0 \leq l \leq L$ and $l' = L + 1$: Since $\alpha_{l-1} \geq \alpha_{\min} - \frac{\varepsilon}{4}$ and $\alpha'_L \geq \alpha'_{\min} + 1 + 2\varepsilon$, we obtain

$$|\langle x_l, y_{L+1} \rangle| \leq A^2 \sum_{j \in \mathbb{N}_0} 2^{(1 - \alpha_{l-1} - \alpha'_L)j} \leq A^2 \sum_{j \in \mathbb{N}_0} 2^{-\frac{7\varepsilon}{4}j}.$$

In the end, $\langle x, y \rangle$ is bounded on U , the bound depending only on A and ε (that is, only on y). This proves that the linear form $x \mapsto \langle x, y \rangle$ is continuous.

□

4.3 Strong topology on $(S^\nu)'$

In the previous theorem, the dual of S^ν has been algebraically identified to a union of spaces $S^{\nu'_\varepsilon}$, for $\varepsilon > 0$, or equivalently to a countable union of spaces

$S^{\nu'_m}$ for $\nu'_m := \nu'_{\varepsilon_m}$, $\varepsilon_m \searrow 0$. As such, it can be endowed with the inductive limit topology on this union, now written $\text{ind}_m S^{\nu'_m}$. We shall now see that, at least when the convexity index (15) $p_0 = 1$, this topology is actually the same as the strong topology on the dual (then written $(S^\nu)'_b$ in the standard notation), that is, the topology of uniform convergence on the bounded sets of S^ν .

Before that, recalling that a Montel space is a barrelled tvs in which every bounded set is relatively compact, we give another remarkable property of S^ν .

Proposition 4.2 *If $p_0 = 1$, then S^ν is a Fréchet-Montel space.*

Proof. In [3, Proposition 6.2] we obtained the following characterization: A subset K of S^ν is compact if and only if it is closed and bounded for each of the distances $d_{\alpha_n, \nu(\alpha_n) + \varepsilon_m}$ (cf. (6), (8)). The special structure of the topology of S^ν show that the latter condition is in fact equivalent to saying that K is bounded in S^ν . So, when $p_0 = 1$, the tvs S^ν is a Fréchet space in which the bounded sets are relatively compact. \square

Corollary 4.3 *S^ν is reflexive if and only if $p_0 = 1$.*

Proof. Every Fréchet-Montel space is reflexive; conversely a reflexive space is by definition necessarily locally convex. \square

Theorem 4 *If $p_0 = 1$, then topologically $(S^\nu)'_b = \text{ind}_m S^{\nu'_m}$.*

Proof. The fact that the canonical injection $S^{\nu'_m} \rightarrow (S^\nu)'_b$ is continuous for every m is obtained using the characterization of the bounded sets of S^ν and a part of the proof leading to the algebraic description of the dual. Indeed, if m is fixed and if B is a bounded set of S^ν , we replace the unit balls U_l by

balls of radius $R > 0$ so that $B \subset U := \bigcap_{l=-1}^L U_l$ (with L depending on m). The same proof then shows that $\langle x, y \rangle \rightarrow 0$ when $y \rightarrow 0$ in $S^{\nu'_m}$, uniformly on B . This proves that the inductive limit topology is stronger than the strong topology (and as a consequence, the inductive limit is Hausdorff).

To prove that the topologies are in fact equivalent, we use the closed graph theorem of De Wilde (see [10] or [12]) for the identity map from the strong dual into the inductive limit: the strong dual is ultrabornological (since it is the strong dual of a Fréchet-Montel space) and the inductive limit is a webbed space. Since the identity has a closed graph, it is continuous. \square

So far the case $p_0 < 1$ remains an open problem (the missing point is to show that the strong dual is ultrabornological or Baire, in order to be able to apply the closed graph theorem).

4.4 Dual of an intersection of Besov spaces

We conclude with an application to a particular case. As we recalled earlier, when ν is concave we have

$$S^\nu = \bigcap_{\varepsilon > 0, p > 0} b_{p,\infty}^{\eta(p)/p-\varepsilon}$$

with (5) $\eta(p) := \inf_{\alpha \geq \alpha_{\min}} (\alpha p - \nu(\alpha) + 1)$. If we invert this Fenchel-Legendre transform we obtain

$$\nu(\alpha) = \inf_{p > 0} (\alpha p - \eta(p) + 1). \quad (28)$$

In that case the dual profile ν' is convex on $[\alpha'_{\min}, \alpha'_{\max})$ and can be directly computed from η as shown in Proposition 4.5 below. Then by Theorems 3 and 4 we know all about the strong topological dual of this intersection of Besov spaces.

We shall keep in mind that the concavity of ν implies that it is now continuous and that the right derivative $\partial^+ \nu(\alpha)$ exists for all $\alpha \geq \alpha_{\min}$.

Lemma 4.4 *If $\nu(\alpha) = \alpha p - \eta(p) + 1$ for some $p > 0$, then $\partial^+ \nu(\alpha) \leq p$.*

Proof. Let $h > 0$, and observe that

$$\nu(\alpha + h) = \inf_{\tilde{p} > 0} ((\alpha + h)\tilde{p} - \eta(\tilde{p}) + 1) \leq (\alpha + h)p - \eta(p) + 1 = \nu(\alpha) + hp$$

and the conclusion follows readily. \square

Proposition 4.5 *If ν is concave and η is its conjugate, then the function η' defined for $p' > 1$ by*

$$\eta'(p') := (p' - 1) \left(1 - \eta \left(\frac{p'}{p' - 1} \right) \right) + 1 \quad (29)$$

is convex and for $\alpha' \in [\alpha'_{\min}, \alpha'_{\max}]$ we have

$$\nu'(\alpha') = \sup_{p' > 1} (\alpha' p' - \eta'(p') + 1). \quad (30)$$

The appearance of (29) should not be surprising if one notices, as an easy consequence of (11) and Hölder's inequality, that when $p > 1$ the dual of $b_{p,1}^{\eta(p)/p}$ is just $b_{p',\infty}^{\eta'(p')/p'}$, with $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Let $p_1, p_2 > 1$ and let $p'_1 := \frac{p_1}{p_1 - 1}$, $p'_2 := \frac{p_2}{p_2 - 1}$. Let $p' := \frac{p'_1 + p'_2}{2}$, and $p'' := \frac{p'}{p' - 1} = \frac{p_2 - 1}{p_1 + p_2 - 2} p_1 + \frac{p_1 - 1}{p_1 + p_2 - 2} p_2$. We have

$$\eta'(p') = (p' - 1)(1 - \eta(p'')) + 1$$

using the concavity of η ,

$$\begin{aligned}
&\leq (p' - 1) \left(1 - \frac{p_2 - 1}{p_1 + p_2 - 2} \eta(p_1) - \frac{p_1 - 1}{p_1 + p_2 - 2} \eta(p_2) \right) + 1 \\
&\leq \frac{p' - 1}{p_1 + p_2 - 2} ((p_2 - 1)(1 - \eta(p_1)) + (p_1 - 1)(1 - \eta(p_2))) + 1 \\
&\leq \frac{p'_1 - 1}{2} (1 - \eta(p_1)) + \frac{p'_2 - 1}{2} (1 - \eta(p_2)) + 1 \\
&\leq \frac{\eta'(p'_1) + \eta'(p'_2)}{2}
\end{aligned}$$

so η' is indeed convex.

Let us now prove (30). We start from the definition (22) of $\nu'(\alpha')$, having fixed an $\alpha' \in [\alpha'_{\min}, \alpha'_{\max})$. Then $\{\alpha : \nu(\alpha) - \alpha > \alpha'\} \neq \emptyset$ (because its infimum is finite). Let us write $\tilde{\alpha} := \inf \{\alpha : \nu(\alpha) - \alpha > \alpha'\}$ and consider two cases.

If $\nu(\tilde{\alpha}) - \tilde{\alpha} > \alpha'$: Necessarily $\tilde{\alpha} = \alpha_{\min}$ (otherwise a contradiction is easily reached using the continuity of ν) and $\nu'(\alpha') = \alpha' - \alpha'_{\min} = \alpha' + \tilde{\alpha}$. The concavity of ν implies that as soon as $p \geq \partial^+ \nu(\tilde{\alpha})$, $\alpha \mapsto \alpha p - \nu(\alpha) + 1$ is increasing on $[\tilde{\alpha}, \infty)$ and (5) becomes $\eta(p) = \tilde{\alpha} p - \nu(\tilde{\alpha}) + 1$. This means that if $p' = \frac{p}{p-1}$ is close enough to 1, $\eta'(p') = (p' - 1)\nu(\tilde{\alpha}) - \tilde{\alpha} p' + 1$ and $\alpha' p' - \eta'(p') + 1 = p'(\alpha' + \tilde{\alpha} - \nu(\tilde{\alpha})) + \nu(\tilde{\alpha})$. The function $p' \mapsto \alpha' p' - \eta'(p') + 1$, being concave, is thus strictly decreasing on $(1, \infty)$ and the supremum on the right-hand side of (30) can be computed as

$$\begin{aligned}
\sup_{p' > 1} (\alpha' p' - \eta'(p') + 1) &= \lim_{p' \rightarrow 1} \alpha' p' - \eta'(p') + 1 \\
&= \alpha' + \tilde{\alpha}.
\end{aligned}$$

If $\nu(\tilde{\alpha}) - \tilde{\alpha} = \alpha'$: Pick any α'' such that $\nu(\alpha'') - \alpha'' > \alpha'$ and observe that $\nu(\alpha'') - \nu(\tilde{\alpha}) > \alpha'' - \tilde{\alpha}$; by concavity again this implies that $\partial^+ \nu(\alpha) > 1$ when $\alpha > \tilde{\alpha}$ is close enough. Thus by Lemma 4.4, for these α , the infimum

in (28) is reached for a $p > 1$. Another consequence is that $\nu'(\alpha') = \alpha' + \inf \{\alpha : \nu(\alpha) - \alpha \geq \alpha'\}$. It follows that

$$\begin{aligned}
\nu'(\alpha') &= \alpha' + \inf \left\{ \alpha : \inf_{p>1} (\alpha p - \eta(p) + 1) - \alpha \geq \alpha' \right\} \\
&= \alpha' + \inf \{ \alpha : \forall p > 1, \alpha(p-1) - \eta(p) + 1 \geq \alpha' \} \\
&= \alpha' + \inf \left\{ \alpha : \forall p > 1, \alpha \geq \frac{\alpha' + \eta(p) - 1}{p-1} \right\} \\
&= \alpha' + \sup_{p>1} \left(\frac{\alpha' + \eta(p) - 1}{p-1} \right).
\end{aligned}$$

On the other hand, changing the variable p' into $p := \frac{p'}{p'-1}$ yields

$$\begin{aligned}
\sup_{p'>1} (\alpha' p' - \eta'(p') + 1) &= \sup_{p>1} \left(\frac{\alpha' p}{p-1} - \frac{1 - \eta(p)}{p-1} \right) \\
&= \alpha' + \sup_{p>1} \left(\frac{\alpha' + \eta(p) - 1}{p-1} \right)
\end{aligned}$$

and the proposition is proved. □

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